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Floquet Phases of Matter via Classical Prethermalization

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We demonstrate that the prethermal regime of periodically driven (Floquet), classical many-body systems can host nonequilibrium phases of matter. In particular, we show that there exists an effective Hamiltonian that captures the dynamics of ensembles of classical trajectories despite the breakdown of this description at the single trajectory level. In addition, we prove that the effective Hamiltonian can host emergent symmetries protected by the discrete time-translation symmetry of the drive. The spontaneous breaking of such an emergent symmetry leads to a subharmonic response, characteristic of time crystalline order, that survives to exponentially late times in the frequency of the drive. To this end, we numerically demonstrate the existence of classical prethermal time crystals in systems with different dimensionalities and ranges of interaction. Extensions to higher order and fractional time crystals are also discussed.

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Many-body Floquet systems can host a variety of intrinsically nonequilibrium phases of matter [1–8]. One of the central challenges in stabilizing such phases is the presence of Floquet heating—a generic interacting system will absorb energy from the driving field until it approaches a featureless, infinite temperature state [9–13]. In quantum systems, strong disorder can induce many-body localization that prevents Floquet heating and enables the system to remain in a nonequilibrium steady state until arbitrarily late times [13–16]. Since localization relies upon the discreteness of energy levels, this specific approach is intrinsically quantum mechanical and naturally begs the following question: To what extent do Floquet nonequilibrium phases require either quantum mechanics or disorder [17–25]?

An elegant but partial answer to this question is provided within the framework of Floquet prethermalization in disorder-free systems [26–39]. When the driving frequency, ω, is larger than the system's local energy scale, J_{local}, Floquet heating is suppressed until exponentially late times, $\tau_{\text{heat}} \sim e^{\omega/J_{\text{local}}}$. In particular, directly absorbing energy from the drive is highly off-resonant, and heating only occurs via higher order processes that involve multiple, correlated local rearrangements. This simple physical intuition holds for both quantum and classical systems.

In the quantum setting, Floquet prethermalization has an additional feature: There exists an effective Hamiltonian that accurately captures the dynamics of the system until τ_{heat}. Whenever the periodic drive induces an emergent symmetry in this effective Hamiltonian, novel nonequilibrium prethermal phases of matter, such as discrete time crystals or Floquet symmetry-protected topological phases, can emerge [1–8,39–47]. Whether analogous phases are also possible in classical many-body systems is significantly more subtle; in particular, although classical prethermalization features slow Floquet heating, there is no effective Hamiltonian that accurately captures the prethermal dynamics [36].

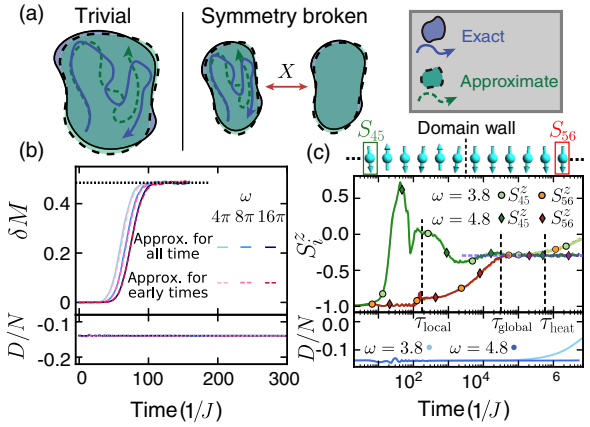

FIG. 1. (a) Schematic depicting trajectories in a classical phase space. The exact Floquet trajectory (blue) diverges from the approximate trajectory under the effective Hamiltonian (green). However, the exact evolution of a *finite region* in phase space is well-captured by the effective Hamiltonian. (b) The dynamics of the magnetization difference, $\delta M(t)$, and the energy density, D/N, for a single initial state with $N = 10^4$. Solid lines depict approximate evolution under D for all times. Dashed lines indicate approximate evolution under D for short times ($t \leq 1/J$), followed by exact Floquet evolution. Agreement between solid and dashed curves highlights the role of classical chaos in the growth of errors. While errors in local observables [i.e., $\delta M(t)$] accumulate rapidly, the energy density remains conserved throughout the dynamics. (c) The prethermal dynamics of an ensemble of initial states quickly converge with increasing frequency. Before Floquet heating brings the system to infinite temperature, the magnetization approaches the value associated with the corresponding prethermal ensemble of D (blue dashed line, computed via Monte Carlo [48]).

In this Letter, we show that the lack of an effective Hamiltonian does not preclude the existence of novel, nonequilibrium phases in classical Floquet systems; we highlight this by explicitly constructing a classical prethermal discrete time crystal (CPDTC). Our main results are threefold. First, we demonstrate that the inability of an effective Hamiltonian to generate the Floquet dynamics is a direct consequence of classical chaos—small errors at early times lead to exponentially diverging single trajectories. This connection to chaos suggests that one should forgo the focus on individual trajectories and rather ask whether there is an effective Hamiltonian that captures the prethermal dynamics of an *ensemble* of trajectories (Fig. 1). We show that this is indeed the case. Second, we prove that, much like the quantum case, the effective Hamiltonian can host an emergent symmetry that is protected by the discrete time translation symmetry of the periodic drive. Finally, we propose, analyze, and numerically simulate a variety of different classical prethermal time crystals in one and two dimensions.

Prethermalization in classical dynamics.—Consider a classical Floquet Hamiltonian, $H_F(t) = H_F(t+T)$, with period $T = 2\pi/\omega$. For $\omega \gg J_{\text{local}}$, one can construct a perturbative expansion of the Floquet dynamics in powers of J_{local}/ω [49]. In general, this Floquet-Magnus expansion diverges, reflecting the many-body system's late-time approach to infinite temperature (via energy absorption from the drive). However, when truncated at an appropriate order, $n^* \sim \omega/J_{\text{local}}$, the expansion defines a static Hamiltonian, D, that remains quasiconserved for exponentially long times (under the full Floquet dynamics) [27,29,36]:

$$\frac{1}{N}|D(t=mT) - D(t=0)| < mJ_{\text{local}} \cdot \mathcal{O}(e^{-\omega/J_{\text{local}}}), \qquad (1)$$

where N is the system size and $m \in \mathbb{N}$ is the number of Floquet cycles. To this end, Eq. (1) precisely formalizes the existence of an intermediate, prethermal regime. In particular, for times $t < \tau_{\text{heat}} \sim \mathcal{O}(e^{\omega/J_{\text{local}}})$, the energy density of the system (measured with respect to D) remains constant up to $\sim \mathcal{O}(J_{\text{local}})$.

Nevertheless, the question remains: Is D also the effective prethermal Hamiltonian that generates the dynamics before τ_{heat}? In the quantum setting, the answer is yes [31,32,41,50]. However, in classical systems, D is only proven to faithfully reproduce the Floquet evolution over a *single* driving period [36]:

$$|O(T) - O'(T)| \leq \mathcal{O}(e^{-\omega/J_{\text{local}}}). \qquad (2)$$

Here, O is a generic local observable and $O(T)$ represents its evolution under the full Floquet Hamiltonian [i.e., $H_F(t)$], while $O'(T)$ represents its evolution under D.

Naively, one might expect the single period errors in Eq. (2) to accumulate additively as one evolves to later times. However, this does not account for compounding effects, where early-time errors propagate through the many-body system and induce additional deviations. In the quantum case, the existence of Lieb-Robinson bounds constrains the propagation of errors and enables one to prove that deviations grow algebraically in the number of Floquet cycles: $|O(mT) - O'(mT)| \leq m^p \mathcal{O}(e^{-\omega/J_{\text{local}}})$; this immediately indicates that D is indeed the effective prethermal Hamiltonian [30–33,41]. In contrast, classical systems exhibit no such bounds—chaos causes the exponential divergence of nearby trajectories, suggesting that errors can in principle accumulate exponentially quickly.

To sharpen this intuition, we numerically explore the Floquet dynamics of a generic classical spin model [51]:

$$H_F(t) = \begin{cases} \sum_{i,j} J_z^{i,j} S_i^z S_j^z + \sum_i h_z S_i^z & 0 \leq t < \frac{T}{3} \\ \sum_i h_y S_i^y & \frac{T}{3} \leq t < \frac{2T}{3} \\ \sum_{i,j} J_x^{i,j} S_i^x S_j^x + \sum_i h_x S_i^x & \frac{2T}{3} \leq t < T \end{cases}, \qquad (3)$$

where $\vec{S}_i$ is a three-dimensional unit vector. Spin dynamics are generated by Hamilton's equations of motion $\dot{S}_i^\mu = \{S_i^\mu, H(t)\}$, using the Poisson bracket relation $\{S_i^\mu, S_j^\nu\} = \delta_{ij}\epsilon^{\mu\nu\rho}S_i^\rho$. The classical dynamics of an observable O are then given by $O(t) = \mathcal{T} e^{\int_0^t L(t')dt'}[O]$, where the superoperator $L[\cdot]$ is defined by $L[\cdot] = \{\cdot, H_F\}$ [52]. At lowest order in the Floquet-Magnus expansion, the static Hamiltonian is given by

$$D = \frac{1}{3}\left(\sum_{i,j} J_z^{i,j} S_i^z S_j^z + J_x^{i,j} S_i^x S_j^x + \vec{h} \cdot \vec{S}_i\right) + \mathcal{O}\left(\frac{1}{\omega}\right). \qquad (4)$$

To investigate the accumulation of errors, we compare the dynamics of local observables evolving under $H_F(t)$ and D in a one-dimensional spin chain ($N = 10^4$) with nearest neighbor interactions [53]. Deviations from the exact Floquet dynamics are measured by computing the magnetization difference between the two trajectories: $\delta M(t) = 1 - (1/N)\sum_i \vec{S}_i(t) \cdot \vec{S}_i'(t)$. As depicted in Fig. 1(b) (top panel), $\delta M(t)$ quickly increases to a plateau value consistent with the spins in the two trajectories being completely uncorrelated; thus, D cannot be thought of as the effective prethermal Hamiltonian for $H_F(t)$. By contrast, the energy density remains conserved throughout the time evolution [bottom panel, Fig. 1(b)], demonstrating slow Floquet heating.

In order to pinpoint the role of chaos in the dynamics of $\delta M(t)$, we consider a slightly modified trajectory; in particular, starting with the same initial state, we first evolve under D for a few Floquet cycles and then under $H_F(t)$ for all subsequent times. Comparing to the exact Floquet dynamics (i.e., evolution under $H_F(t)$ for all times), this protocol only differs at very early times.

Indeed, beyond an initial, exponentially small difference in the trajectories [arising from Eq. (2)], any additional deviation solely arises from the chaotic compounding of errors. As depicted in Fig. 1(b) (dashed curves), the magnetization difference between the modified trajectory and that of the exact Floquet dynamics tracks $\delta M(t)$ for all times. Crucially, this agreement demonstrates that chaos dominates the growth of $\delta M(t)$ and prevents D from being the effective prethermal Hamiltonian.

Prethermal dynamics of trajectory ensembles.—While the evolution of a single trajectory cannot be captured by an effective Hamiltonian, we conjecture that D captures the dynamics of *ensembles* of trajectories [Fig. 1(a)]; by considering an initial state composed of a region of phase space (as opposed to a single point), the details of individual chaotic trajectories become "averaged out." This conjecture is made up of two separate components: (i) during the prethermal plateau, the system approaches the canonical ensemble of D, and (ii) D accurately captures the dynamics of observables as the system evolves from local to global equilibrium. This last component highlights the two stage approach to the prethermal canonical ensemble. First, observables on nearby sites approach the same value and the system *locally* equilibrates (this occurs at time τ_{local}). Afterward, the system becomes globally homogeneous as it approaches global equilibrium at time τ_{global}.

To investigate these components, we implement the following numerical experiment: Starting from an $N = 100$ spin chain, we construct an ensemble of initial states with a domain wall in the energy density at the center of the chain and study the Floquet dynamics of the local magnetization S_i^z and energy density D/N [Fig. 1(c)] [54].

Focusing on the late-time regime (but before Floquet heating), we find that the magnetization on opposite sides of the domain wall approaches the *same prethermal plateau* [Fig. 1(c)]; this precisely corresponds to the global equilibration of our spin chain. Crucially, the value of this plateau *quantitatively* agrees with the mean magnetization of the corresponding canonical ensemble of D calculated at the same energy density via Monte Carlo [Fig. 1(c)] [48]. Notably, we find agreement not only with the average value but also with the entire distribution [48], thus verifying the first component of the conjecture.

To investigate the second component, we time evolve the same ensemble of initial states for different frequencies of the drive [55]. So long as $\tau_{\text{heat}} \gg \tau_{\text{global}}$, we find that the dynamics of local observables rapidly converge as a function of increasing frequency [Fig. 1(c)]. Since the $\omega \to \infty$ limit of $H_F(t)$ precisely corresponds to Trotterized evolution under D, the convergence observed in Fig. 1(c) indicates that D is indeed the prethermal Hamiltonian for trajectory ensembles. This is in stark contrast to the dynamics of a single trajectory, where local observables fail to converge with increasing frequency [48].

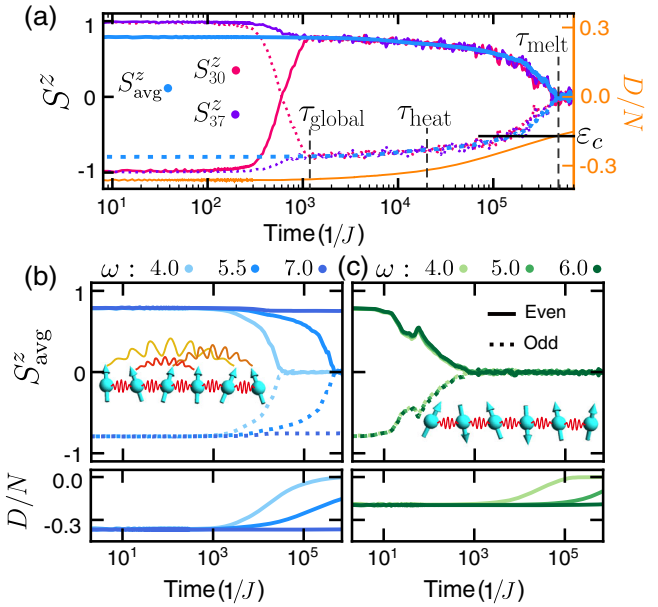

FIG. 2. (a) Dynamics of a classical prethermal time crystal in a one-dimensional long-range interacting spin chain. At τ_{global}, different sites exhibit the same magnetization, indicating equilibration. For an exponentially long intermediate time window, $\tau_{\text{global}} < t < \tau_{\text{melt}}$, the system oscillates between positive and negative magnetization values for even (solid line) and odd (dotted line) periods. This subharmonic response remains stable until the energy density crosses ε_c and the CPDTC melts. (b),(c) Prethermal dynamics of the spin chain for different frequencies ω with either long-range, (b), or short-range, (c), interactions. For long-range interactions, the lifetime of the CPDTC is exponentially enhanced by increasing the frequency of the drive. For short-range interactions, transient period doubling decays at a frequency independent timescale, which is significantly shorter than the Floquet heating time (bottom panel).

Interestingly, however, even for a single trajectory, the Floquet dynamics of either *spatially* or *temporally* averaged quantities are well captured by D. The intuition is simple: by averaging over different times or different spatial regions, a single trajectory effectively samples over an ensemble of different configurations [Fig. 1(a)]. This insight yields a particularly useful consequence, namely, that the dynamics of a *single trajectory* already encode the prethermal properties of the many-body system.

Prethermal dynamics with symmetry breaking.—Throughout our previous discussions, energy conservation is the only constraint that restricts the many-body dynamics within phase space. However, symmetry breaking can lead to additional constraints; for example, if D exhibits a discrete symmetry and this symmetry is broken at low energy densities, then phase space is naturally split into multiple disjoint regions corresponding to different values of the order parameter. As a result, the many-body dynamics under D are restricted to one such region.

Floquet evolution complicates this story. In particular, one might worry that the micromotion of the Floquet dynamics

could move the system between different symmetry-broken regions of phase space. If this were the case, prethermal symmetry-breaking phases would not be stable. Fortunately, the ability of D to approximate the dynamics over a single period [i.e., Eq. (2)], is sufficient to constrain the Floquet evolution to a specific symmetry-broken region.

To see this, consider, for example, a system where D exhibits a discrete $\mathbb{Z}_2$ symmetry and hosts a ferromagnetic phase whose order parameter is given by the average magnetization. When the energy density is below the critical value, the magnetization of the system can either be S^z_{avg} or $-S^z_{\text{avg}}$. Given energy conservation, under a single period of evolution, the magnetization must remain the same or change sign. However, Eq. (2) guarantees that the time evolved magnetization density can change, at most, by an exponentially small value in frequency. This ensures that for sufficiently large driving frequencies, the magnetization cannot change sign (i.e., move to the other symmetry-broken region) and the prethermal ferromagnet remains stable.

Crucially, symmetries of D can have two different origins: They can be directly inherited from $H_F(t)$, or they can emerge as a consequence of the time translation symmetry of the drive [40,41]. In the latter case, this can give rise to intrinsically nonequilibrium phases of matter. To date, the study of such nonequilibrium prethermal phases has been restricted to quantum systems [39,56–63], where one can explicitly prove their stability [40,41]. Here, we generalize and extend this analysis to classical many-body spin systems by taking the large-S limit of the quantum dynamics [36,48].

Consider a Floquet Hamiltonian that is the sum of two terms, $H_F(t) = H_X(t) + H_0(t)$. During a single driving period, $H_X(t)$ generates a global rotation $X[\cdot] = \mathcal{T} e^{\int_0^T \{\cdot, H_X(t)\} dt}$ such that the system returns to itself after M periods (i.e., $X^M[\cdot] = \mathbb{I}[\cdot]$, where $\mathbb{I}$ is the identity map). $H_0(t)$ captures the remaining interactions in the system [52]. For sufficiently large frequencies, the single period dynamics (in a slightly rotated frame) are accurately captured by $X \circ e^{T\{\cdot, D\}}$, where D is obtained via a Magnus expansion in the toggling frame [48]; this expansion guarantees that the dynamics generated by D commute with X and thus X generates a discrete $\mathbb{Z}_M$ symmetry of the effective Hamiltonian [40,41]. Indeed, at lowest order, D is simply given by the time-independent terms of $H_0(t)$ that are invariant under the global rotation.

The resulting prethermal Floquet dynamics are most transparent when analyzed at stroboscopic times $t = mT$ in the toggling frame of the X rotations, wherein an observable O becomes $\widetilde{O}(mT) = X^{-m}[O(mT)]$. In this context, the dynamics of $\widetilde{O}$ are simply generated by D, i.e., $\widetilde{O}(mT) = e^{mT\{\cdot, D\}}[\widetilde{O}(t=0)]$. Thus, if the emergent $\mathbb{Z}_M$ symmetry of D becomes spontaneously broken, the system will equilibrate to a thermal ensemble of D with a nonzero order parameter.

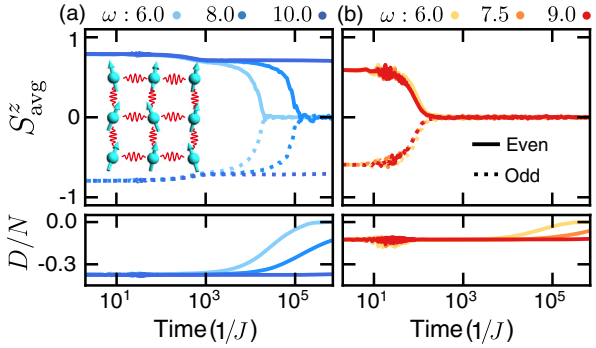

FIG. 3. Prethermal dynamics of a nearest-neighbor interacting classical spin model on the square lattice. (a) For a low-energy-density initial state, the system exhibits robust period doubling until exponentially late times. (b) For a high-energy-density initial state, the magnetization decays to zero rapidly, well before the Floquet heating time. This highlights the presence of a critical energy density and the importance of symmetry breaking for the existence of a CPDTC.

In the lab frame, the dynamics of O are richer: The global rotation changes the order parameter every period, only returning to its original value after M periods. As a result, the system exhibits a subharmonic response at frequencies $1/(MT)$ [40,41]. This is precisely the definition of a classical prethermal discrete time crystal.

Building a CPDTC.—Let us now consider the Floquet Hamiltonian in Eq. (3) with an additional global π rotation around the $\hat{x}$ axis at the end of each driving period [64]. At leading order, X corresponds to the global π rotation, while D is given by the time averaged terms of $H_F(t)$ that remain invariant under X (i.e., Eq. (4) with $h_y = h_z = 0$). To this end, we will use the energy density D/N and the average magnetization S^z_{avg} to diagnose the prethermal dynamics and the CPDTC phase.

Let us begin by considering a one-dimensional system with long-range interactions $J_z^{i,j} = J_z |i-j|^{-\alpha}$; when $\alpha \leq 2$, D exhibits ferromagnetic order below a critical temperature (or, equivalently, a critical energy density ε_c that can be determined via Monte Carlo calculations) [65,66]. Taking $\alpha = 1.8$ and $N = 320$, we compute the Floquet dynamics starting from an ensemble with energy below ε_c [Fig. 2(a)] [67]. After the initial equilibration to the prethermal state ($t \gtrsim \tau_{\text{global}}$), the magnetization becomes homogeneous across the entire chain, signaling equilibration with respect to D [68]. Crucially, as depicted in Fig. 2(a), throughout this prethermal regime, the magnetization exhibits robust period doubling, taking on positive values at even periods and negative values at odd periods. This behavior remains stable until the CPDTC eventually "melts" at an exponentially late time, τ_{melt}, when the energy density crosses the critical value ε_c of the ferromagnetic transition of D [Fig. 2(a)].

A few remarks are in order. First, because τ_{heat} is significantly longer than the interaction timescale, the

system evolves between different thermal states of D as it absorbs energy from the drive. Second, the lifetime of the CPDTC is controlled by the Floquet heating rate and thus the frequency of the drive. Indeed, by increasing ω, the lifetime of the CPDTC is exponentially enhanced, while the global equilibration time remains constant [Fig. 2(b)] [48]. Third, we emphasize that the observed CPDTC is fundamentally distinct from period-doubling bifurcations in classical dynamical maps (e.g., the logistic map) or the subharmonic response of a parametrically driven nonlinear oscillator [21,69–85]. In particular, it occurs in an *isolated many-body classical system with conservative dynamics*.

Let us conclude by highlighting the central role of spontaneous symmetry breaking in observing the CPDTC. We do so by controlling the range of interactions, the dimensionality, and the energy density of the initial ensemble. To start, we consider the short-ranged version (i.e., nearest neighbor interactions) of the 1D classical spin chain discussed above. Without long-range interactions, ferromagnetic order is unstable at any finite temperature [86], and this immediately precludes the existence of a CPDTC. This is indeed borne out by the numerics [Fig. 2(c)]: We observe a fast, frequency-independent decay of the magnetization to its infinite-temperature value.

While nearest neighbor interactions cannot stabilize ferromagnetism in 1D, they do so in higher dimension. To this end, we explore the same Floquet model [i.e., Eq. (3)] on a two-dimensional square lattice. For sufficiently low energy densities, the system equilibrates to a CPDTC phase [Fig. 3(a)], while above the critical temperature, the system equilibrates to a trivial phase [Fig. 3(b)]. We hasten to emphasize that our framework is not restricted to the period-doubled ($M = 2$) CPDTC and it immediately ports over to more general notions of time crystalline order, including both higher order ($M > 2$) and *fractional* CPDTCs (see the Supplemental Material for additional numerics) [24,48].

Our work opens the door to a number of intriguing directions. First, it would be interesting to explore the generalization of classical prethermal time crystals to quasiperiodic driving [87]. Second, although we have presented extensive numerical and analytic evidence for the presence of an effective Hamiltonian (for trajectory ensembles), sharpening our analysis into a proof would provide additional insights in the nature of many-body classical Floquet systems.

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Note added.—During the completion of this work, we became aware of complementary work exploring classical prethermal phases of matter [88,89].

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[49] More precisely, we note that the classical Floquet dynamics are generated by the superoperator $L(t)[\cdot] = \{\cdot, H_F(t)\}$. The time evolution operator over a single period is then given by $U_F = \mathcal{T} e^{\int_0^T L(t)dt} \equiv e^{L_F T}$, where L_F is a time-independent superoperator. The static Hamiltonian, $\mathcal{H}_F$, corresponding to L_F is then given by $L_F = \{\cdot, \mathcal{H}_F\}$. The Floquet-Magnus expansion constructs $\mathcal{H}_F$ order by order in $(J_{\text{local}}/\omega)$, i.e., $\mathcal{H}_F = \sum_{n=0}^{\infty} (J_{\text{local}}/\omega)^n D^{(n)}$, where $D^{(n)}$ is the nth order term of the expansion. Note

that the effective static Hamiltonian, D, is then defined as $D = \sum_{n=0}^{n^*} (J_{\text{local}}/\omega)^n D^{(n)}$.

[50] Assuming that the system is extensive and power-law light cones exist as defined via Lieb-Robinson bounds.

[51] Throughout this Letter, we use the following generic set of parameters: $\{J_z, J_x, h_x, h_y, h_z\} = \{-1.0, 0.79, 0.17, 0.23, 0.13\}$.

[52] We note that the multiplication of the superoperators (functions of observables) should be understood as function composition. In particular, $(L_1 \circ L_2)[\cdot] = L_1(L_2[\cdot])$. The nth power of L is then defined inductively by $L^n = L \circ L^{n-1}$. Therefore, evolving under the Floquet Hamiltonian for m periods, an observable becomes $O(mT) = (\mathcal{T} e^{\int_0^T L(t)dt})^m [O]$. In a similar fashion, let us define X^{-1} as the inverse of the map X.

[53] In general, the chaotic nature of D means that numerically integrating the equations of motion to later times requires exponentially better precision, making the numerical treatment very difficult. By contrast, each term of $H_F(t)$ corresponds to a precession of the spins along one of three axes, which can be straightforwardly analytically calculated without resorting to numerical integration methods.

[54] We initialize each spin along either $+\hat{z}$ or $-\hat{z}$ direction. By tuning the number of domain walls, we can control the local energy density of the system. While the spins on the right half of the chain are initialized in a completely ferromagnetic state, the spins on the left half repeat the following pattern: $\downarrow \uparrow \downarrow$. Therefore, the energy density across the chain exhibits a domain wall at the center of the chain. To bring out the ensemble effect, we add small random noise to the azimuthal angle and perform an average of the subsequent dynamics over these slightly different initial states.

[55] The local equilibration time τ_{local} corresponds to the time when nearby spins approach the same value. To identify τ_{local}, we measure the time when S^z_{43} and S^z_{45}, initially pointing in opposite directions, exhibit the same magnetization.

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[64] The π rotation can be generated by a magnetic field along the $\hat{x}$ direction for time τ_π with strength $\pi/2\tau_\pi$. Since the dynamics are independent of the value of τ_π chosen, we consider it to be an instantaneous rotation, $\tau_\pi \to 0$. In order to better highlight the role of D throughout the prethermal regime, we consider instead a symmetrized version of $H_F(t)$ such that its first-order contribution in inverse frequency is zero and thus the dynamics are better captured by the *zero*th order terms. We note that this does not change the results qualitatively.

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[67] To prepare the initial state ensemble, we first start with the fully polarized system, flip every tenth spin, and then add a small amount of noise to the azimuthal angle. The energy density of the resulting ensemble is $\varepsilon = -1.10$.

[68] We note that the fluctuations of S^z_i are local thermal fluctuations and that they decrease as we average over more realizations.

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